BENDING OF A STRIP

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The problem of bending of a half-strip rigidly fastened along the short edge is examined. An integral equation is constructed for the normal stress at the clamping and the character of singularities of its solutions at the corners is investigated. By the method of collocation the given equation is reduced to a system of linear algebraic equations.

Numerical calculations were carried out for the case of bending of the half-strip by a moment applied at infinity.

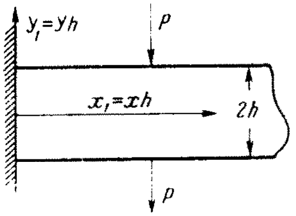

Fig. 1

1. Let us examine the problem of bending of a half-strip under the following boundary conditions (Fig. 1):

$$u = v = 0, \qquad x = 0, \qquad |y| \leqslant 1 \tag{1.1}$$

$$\sigma_{y1} = g(x)\,\mathrm{sgn}\,y, \qquad \tau_{x1y1} = r(x), \qquad y = \pm 1 \tag{1.2}$$

Here u, v are displacements along axes x_1, y_1, respectively, and τ_{x1y1}, σ_{y1} are the tangential and normal stresses. For derivation of the integral equation for the normal stress $\sigma(t)$ at the clamping we will make use of the method developed in paper [1]. Then we obtain

$$\int_{-1}^{1} \sigma(t)\left\{\ln|y-t| + \frac{2\nu}{(\nu+2)}(1-t)\left[\frac{(1+y)}{(2+y-t)^2} - \frac{(1-y)}{(2-y-t)^2}\right] - \right.$$

$$\left. - \frac{\nu^2+2\nu+2}{\nu(\nu+2)}\ln\frac{(2+y-t)}{(2-y-t)}\right\}dt + R(\sigma) + \frac{2}{(\nu+2)}K(y)\int_{-1}^{1}\sigma(t)\,t\,dt +$$

$$+ \gamma_0\pi y - f(y) = 0, \qquad |y| \leqslant 1 \tag{1.3}$$

$$f(y) = -\frac{4(1+\nu)}{(\nu+2)}\int_0^\infty \bar g(\lambda)\left[\left(\lambda\,\mathrm{sh}\,\lambda - \frac{\mathrm{ch}\,\lambda}{\nu}\right)\varphi(\lambda y) - \mathrm{ch}\lambda\;\lambda y\,\psi(\lambda y)\right]\frac{d\lambda}{\lambda\Delta_-} +$$

$$+ \frac{4(1+\nu)}{(\nu+2)}\int_0^\infty \bar r(\lambda)\left[\left(\lambda\,\mathrm{ch}\,\lambda - \frac{1+\nu}{\nu}\,\mathrm{sh}\,\lambda\right)\varphi(\lambda y) - \lambda y\,\mathrm{sh}\lambda y\,\psi(\lambda y)\right]\frac{d\lambda}{\lambda\Delta_-}$$

$$R(\sigma) = \frac{2}{(\nu+2)}\sum_{s=0}^{\infty} L_s \int_{-1}^{1}\sigma(t)\,t^{2s+3}\,dt$$

$$L_s = \sum_{k=0}^{\infty}(2k+3)\,y^{2k+2}\frac{\Gamma(2m+2k+7)}{\Gamma(2k+4)\,\Gamma(2m+4)}\left(\frac{1}{2}\right)^{2k+2m+5}\left\{\frac{P_{k+m}}{\nu} + \right.$$

$$+\beta_{k+m}+\nu\left[(2k+3)(2m+3)P_{k+m}+\varphi_{k+m}\right]\Big\}$$

$$K(y)=\sum_{m=0}^{\infty}\left(\frac{y}{2}\right)^{2m+3}\left(\frac{\alpha_m}{\nu}+\delta_m+\nu\varepsilon_m\right),\qquad \Gamma(n+1)=n\Gamma(n)$$

$$\nu=\frac{1}{1-2\mu},\quad \varphi(\lambda y)=\mathrm{sh}\lambda y-\lambda y,\quad \psi(\lambda y)=\mathrm{ch}\lambda y-1,\quad \Delta_-=\mathrm{sh}2\lambda-2\lambda$$

$$\bar{g}(\lambda)=\int_0^\infty g(x)\sin\lambda x\,dx,\quad \bar{r}(\lambda)=\int_0^\infty r(x)\cos\lambda x\,dx$$

Here P_n, β_n, φ_n, α_n, δ_n, and ϵ_n are coefficients which are computed once and for all, γ_0 is an arbitrary constant, μ is Poisson's ratio.

Eq. (1.3) is the integral equation of Fredholm of the first kind. The kernel of this equation has a moving singularity on the diagonal $y=t$ and fixed singularities at points $y=\pm 1$. The fixed singularity in the kernel complicates the examination of the equation and its numerical solution.

2. Let us devote our attention to the investigation of the singularity of the solution of the problem under examination. We differentiate equation (1.3) with respect to the variable y, then carry out the substitution $1-t=u$ and $1-y=v$ and take into account that $\sigma(-t)=-\sigma(t)$.

As a result we obtain

$$\int_0^1 q(u)\left[K(u,v)+K(u,2-v)+K_1(u,v)\right]du=f(v)\quad(0\leqslant v\leqslant 2)\tag{2.1}$$

$$q(u)=\sigma(1-u),\quad K(u,v)=\frac{1}{(u-v)}+\frac{2\nu}{(\nu+2)}\frac{u(u-v)}{(u+v)^3}-\frac{\nu^2+2\nu+2}{\nu(\nu+2)}\frac{1}{(u+v)}$$

Here $K_1(u,v)$ and $f(v)$ are continuous functions and continuously differentiable any number of times with respect to u and v in the region of their change. Apparently, the singularity of solution near the point $v=0$ will be determined only by the kernel $K(u,v)$. Let us make use of results of [2] in which it was shown that the solution of the integral Eq.

$$\int_0^\infty q_1(u)K(u,v)\,du=f_1(v)\qquad(0\leqslant v<\infty)\tag{2.2}$$

near the boundary $v=0$ can be represented in the form

$$q_1(v)=\sum_{k=0}^{\infty}M_k v^{p_k-1}\tag{2.3}$$

if $f_1(v)$ is a continuous function with all its derivatives in the vicinity $v=0$. Here p_k are roots of the characteristic Eq.

$$2\varkappa\cos\pi p-4p^2+1+\chi^2=0,\qquad \varkappa=3-4\mu\tag{2.4}$$

The real part of p_k is positive. Eq. (2.4) for any $0<\mu<0.5$ has one real positive root $p_0<1$. The remaining roots are complex and as is shown by specific computations [3]

$$\operatorname{Re}p_k>1.6,\quad k\geqslant 1$$

Since the singularity of the solution of Eq. (2.2) near the boundary $v=0$ is determined only by the behavior of the kernel $K(u, v)$ near this point, the solution of Eq. (2.1) in this vicinity is represented by Eq. (2.3).

On the basis of condition $\sigma(-y) = -\sigma(y)$ we can draw an analogous conclusion about the behavior of the solution near the point $1 + y = 0$.

3. Having examined singularities of the solution of the integral Eq. (1.3) at the ends of section $[-1, 1]$, we select as a numerical method the analog of the method of Multhopp-Kalandi [1]. In accordance with expansion (2.3) the approximate solution of the problem may be sought in the form

$$\sigma(t) = D\left[\frac{(1+t)^{2-p_0}}{(1-t)^{1-p_1}} - \frac{(1-t)^{2-p_1}}{(1+t)^{1-p}}\right] + \sqrt{1-t^2}\sum_{n=0}^{(n=N)} E_n U_{2n+1}(t) \tag{3.1}$$

where $U_{2n+1}(t)$ are Chebyshev functions of the second kind.

We differentiate the left part of Eq. (1.3) with respect to variable y, after that we substitute (3.1) into the obtained equation, then from the condition of equality to zero of the last relationship in the selected nodes of collocation y_k we arrive at a system of linear algebraic equations with respect to unknowns D and E_n.

Table 1

n	$-P_n$	$-\beta_n$	φ_n
0	0.2614406 (−1)	0.1008765	0.1272675
1	0.4906414 (−2)	0.2294624 (−1)	0.4891501 (−1)
2	0.1087015 (−2)	0.605872 (−2)	0.1797972 (−1)
3	0.2570645 (−3)	0.167498 (−2)	0.6316076 (−2)
4	0.625843 (−4)	0.468406 (−3)	0.2131532 (−2)
5	0.154462 (−4)	0.130806 (−3)	0.6952711 (−3)
6	0.383739 (−5)	0.363067 (−4)	0.220422 (−3)
7	0.95640 (−6)	0.10002 (−4)	0.68244 (−4)
8	0.23873 (−6)	0.27358 (−5)	0.20707 (−4)
9	0.5962 (−7)	0.7431 (−6)	0.6173 (−5)
10	0.149 (−7)	0.201 (−6)	0.1816 (−5)
11	0.3725 (−8)	0.538 (−7)	0.5285 (−6)
12	0.9312 (−9)	0.1440 (−7)	0.1520 (−6)
13	0.2328 (−9)	0.3834 (−8)	0.4334 (−7)
14	0.5820 (−10)	0.1017 (−8)	0.1225 (−7)
15	0.1455 (−10)	0.2688 (−9)	0.3440 (−8)

Adopted abbreviation of notation, 0.2614406 (−1) indicates − 0.02614406

Table 2

n	$-\alpha_n$	$-\delta_n$	ε_n
0	1.0387106	3.3196631	−1.8125050
1	0.1568644	0.6052593	−0.2071643 (−1)
2	0.3925131 (−1)	0.1835699	0.1165609
3	0.1087015 (−1)	0.605872 (−1)	0.8196588 (−1)
4	0.3084774 (−2)	0.2009976 (−1)	0.4186040 (−1)
5	0.876180 (−3)	0.6557687 (−2)	0.1845111 (−1)
6	0.247139 (−3)	0.2092894 (−2)	0.7417253 (−2)
7	0.69073 (−4)	0.653520 (−3)	0.279335 (−2)
8	0.19128 (−4)	0.20005 (−3)	0.10015 (−2)
9	0.5252 (−5)	0.60189 (−4)	0.34527 (−3)
10	0.1431 (−5)	0.1783 (−4)	0.1152 (−3)
11	0.388 (−6)	0.523 (−5)	0.375 (−4)
12	0.1043 (−6)	0.1508 (−5)	0.1198 (−4)
13	0.2794 (−7)	0.4321 (−6)	0.3751 (−5)
14	0.74503 (−8)	0.12270 (−6)	0.11559 (−5)
15	0.1979 (−8)	0.34575 (−7)	0.35131 (−6)
16	0.5238 (−9)	0.96769 (−8)	0.10550 (−6)

Apparently the found solution $\sigma(t)$ will be a solution of the initial Eq. (1.3). As nodes of collocation, roots of Chebyshev polynomials of the first kind were selected.

$$y_k = \cos\frac{(2k-1)}{4N}\pi \qquad (k = 1, \ldots, N)$$

Here N is the number of points of division of section [0, 1].

In this manner the problem was solved for the case of bending of the half-strip by the moment M, applied at infinity in the absence of normal and shear stresses on the surfaces of the half-strip ($\bar{g}(\lambda) = 0, \quad \bar{r}(\lambda) = 0$) with the additional condition

$$\int_{-1}^{1} \sigma(t) \cdot t\,dt = \frac{M}{h^2} \tag{3.2}$$

From (3.2) the arbitrary constant γ_0 is determined. Because of the specially selected form of the approximation of solution (3.1) all integrals of the problem are taken in closed form. As a result of calculations a system of linear algebraic equations of the following form was obtained:

Table 3

y	$N = 5$	$N = 6$
0.12050	0.16638	0.16424
0.21901	0.30122	0.29969
0.40849	0.55829	0.56068
0.58168	0.79729	0.79617
0.69671	0.96066	0.95797
0.90045	1.30331	1.30593
0.95534	1.54167	1.53738

Table 4

	$N = 5$	$N = 6$
D	0.21977	0.22060
E_0	0.36917	0.36577
E_1	0.12855 (−1)	0.10304 (−1)
E_2	−0.77423 (−2)	−0.10260 (−1)
E_3	−0.49815 (−2)	−0.79340 (−2)
E_4		−0.38509 (−2)
γ_0	0.31556	0.31230

$$Da_m + \sum_{n=0}^{n=N-2} b_{nm}E_n + \gamma_0 = 0 \qquad (m = 1, \ldots, N) \tag{3.3}$$

$$D\frac{1}{\varepsilon} + E_0\, 0.25 = \frac{1}{\pi}\frac{M}{h^2} \qquad \left(\varepsilon = \frac{3\sin\pi p_0}{4(1-p_0)(2-p_0)(3-2p_0)}\right) \tag{3.4}$$

Eq. (3.4) follows from condition (3.2). In Table 1 values are given for P_n, β_n and ϕ_n. Values of coefficients α_n, δ_n and ϵ_n are presented in Table 2.

In the computation of coefficients presented above their representations through Howlands integrals [4] were utilized.

In Table 3 results of calculations are presented for stresses at the $\sigma^\circ = \sigma(y)\,(h^2/M)$ depending on the number of points of collocation ($N = 5, 6$) for $\mu = 0.31741$ and $p_0 = 0.70000$. It is evident from this Table that the fifth approximation differs from the sixth by no more than 1.5%.

Table 4 contains values of coefficients D and E_n for the fifth and sixth approximations.

From an examination of Tables it follows that the presented method has a high degree of convergence, in this connection the highest degree of accuracy of solution is achieved near the corner points.

Values of stresses found from the equation for strength of materials $\sigma(t) = M/h^2\, 1.5t$, differ from practically exact values obtained in this paper by no more than 10% for $|t| \leq 0.95$.

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BIBLIOGRAPHY

1. Vorovich, I.I., and Kopasenko, V.V., Some problems in the theory of elasticity for a semi-infinite strip. *PMM* Vol. 30, No. 1, 1966.
2. Kurshin, L.M., Mixed plane boundary value problem of the theory of elasticity for a quadrant. *PMM* Vol. 23, No. 5, 1959.
3. Ufliand, Ia.S., Integral Transformations in Problems of the Theory of Elasticity. (p. 141). M.-L. Izd. Akad. Nauk, SSSR, 1963.
4. Nelson, C.W., New Tables of Howlands and Related Integrals. Math. Comp. Vol. 15, (pp. 12 to 18), 1961.

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